HEAT CONDUCTION PROBLEM IN THE CASE OF A BILATERALLY BONDED PLATE WITH HEAT TRANSFER BETWEEN IT AND THE SURROUNDING MEDIUM

A. I. Uzdelev and L. M. Serebryakova

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In solving the heat conduction problem in [1] for a bilaterally bonded plate no account was taken of the heat transfer through the bases. Here, by the method of a small parameter, formulas are derived for the functions of the temperature which take into account the heat transfer through the bases.

1. The outside and the inside contour of an isotropic bilaterally bonded plate are defined by the equations:

$$x = r_i(\cos\Theta + \varepsilon_i \cos n_i\Theta),$$
$$y = r_i(\sin\Theta - \varepsilon_i \sin n_i\Theta) \ (i = 1,2), \ |\varepsilon_i| < 1. \tag{1}$$

Here i = 1 refers to the outside contour and i = 2 refers to the inside contour. A constant temperature is maintained on the outside surface (M_1) and on the inside surface (M_2). The heat exchange with the surrounding medium through the bases of the plate is assumed to follow Newton's law [2]. The ambient temperature is T_0. It is also assumed that the thermophysical properties of the plate material do not depend on the temperature. We will establish the temperature distribution mode in the plate. The equation of heat conduction and the boundary conditions are [3]:

$$\nabla^2 T - \nu^2(T - T_0) = 0, \tag{2}$$

$$T = M_i \text{ on } L_i \ (i = 1,2). \tag{3}$$

Here

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial\Theta^2}; \quad \nu^2 = \frac{2\beta}{\lambda h}.$$

Conditions (3) will be represented in the Cartesian system of coordinates x, y:

$$T(x_i + \varepsilon_i k_i, \ y_i + \varepsilon_i m_i) = M_i \text{ on } L_i. \tag{4}$$

Here $x_i = r_i \cos\Theta$, $y_i = r_i \sin\Theta$, $k_i = r_i \cos n_i\Theta$, $m_i = -r_i \sin n_i\Theta$; x_i, y_i are the coordinates of a point on a circle with radius r_i; $\varepsilon_i k_i$, $\varepsilon_i m_i$ are the increments of coordinates which map points on the contour L_i onto points on the circle with radius r_i. In order to solve the problem, we use the method of a small parameter. As the parameter we choose $\varepsilon = \varepsilon_1$; $\varepsilon_2 = s\varepsilon$. The unknown function is expressed, in the fourth approximation, as

$$T = \sum_{k=0}^{4} \varepsilon^k T_k. \tag{5}$$

Here

$$T_0 = f_0(r), \ T_1 = \varphi_1(r, \alpha_1)\cos\alpha_1\Theta + \varphi_1(r, \alpha_2)\cos\alpha_2\Theta,$$
$$T_2 = f_2(r) + \varphi_2(r, \alpha_1)\cos 2\alpha_1\Theta + \varphi_2(r, \alpha_2)\cos 2\alpha_2\Theta,$$
$$T_3 = \Phi_3(r, \alpha_1)\cos\alpha_1\Theta + \varphi_3(r, \alpha_1)\cos 3\alpha_1\Theta$$

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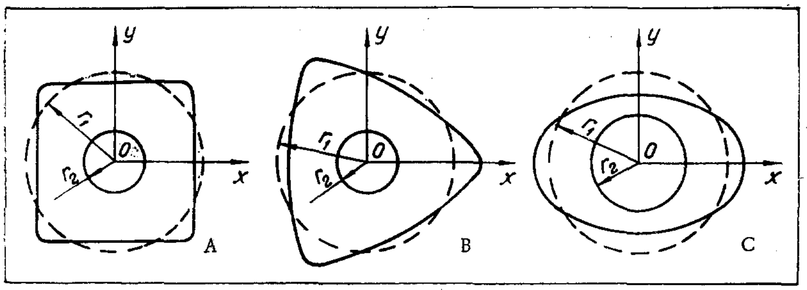

Fig. 1. View of the plates, whose outside contour is defined by Eq. (1), with the following parameters: $r_i = r_1$, $\varepsilon_i = -1/9$, $n_i = 3$ (A); $r_i = r_1$, $\varepsilon_i = 1/5$, $n_i = 2$ (B); $r_i = r_1$, $\varepsilon_i = 1/7$, $n_i = 1$ (C).

$$+\Phi_3(r,\ \alpha_2)\cos\alpha_2\Theta + \varphi_3(r,\ \alpha_2)\cos 3\alpha_2\Theta,$$

$$T_4 = f_4(r) + \Phi_4(r,\ \alpha_1)\cos 2\alpha_1\Theta + \varphi_4(r,\ \alpha_1)\cos 4\alpha_1\Theta$$

$$+\Phi_4(r,\ \alpha_2)\cos 2\alpha_2\Theta + \varphi_4(r,\ \alpha_2)\cos 4\alpha_2\Theta;$$

$$f_k(r) = A_k I_0(\nu r) + B_k K_0(\nu r) \quad (k = 0,\ 2,\ 4),$$

$$\varphi_k(r,\ \alpha) = C_{k\alpha} I_{k\alpha}(\nu r) + D_{k\alpha} K_{k\alpha}(\nu r) \quad (k = 1,\ 2,\ 3,\ 4),$$

$$\Phi_3(r,\ \alpha) = E_\alpha I_\alpha(\nu r) + N_\alpha K_\alpha(\nu r),$$

$$\Phi_4(r,\ \alpha) = F_{2\alpha} I_{2\alpha}(\nu r) + R_{2\alpha} K_{2\alpha}(\nu r);$$

$$\alpha_1 = n_1 + 1;\ \alpha_2 = n_2 + 1.$$

We will now define the boundary conditions for the functions T_k ($k = 0, 1, 2, 3, 4$). For this purpose, the temperature function given at the outside and the inside contour is expanded into Taylor series. Into these series we insert function T defined in (5) and its partial derivatives. Equating the coefficients of like powers in ε, we obtain the conditions at circles with radii $r = r_1$ and $r = r_2$ for determining the integration constants. These conditions are

$$f_0(r_i) = M_i,\ \varphi_1(r_i,\ \alpha_i) = -\delta_i r_i \frac{df_0}{dr},$$

$$f_2(r_i) = -\frac{\delta_i r_i}{4}\left\{\delta_i\left[r_i \frac{d^2 f_0}{dr^2} - (2n_i + 1)\frac{df_0}{dr}\right] + 2\frac{d\varphi_1}{dr}\right\},$$

$$\varphi_2(r_i,\ \alpha_i) = -\frac{\delta_i r_i}{4}\left\{\delta_i\left[r_i \frac{d^2 f_0}{dr^2} + (2n_i + 1)\frac{df_0}{dr}\right] + 2\frac{d\varphi_1}{dr}\right\},$$

$$\Phi_3(r_i,\ \alpha_i) = -\frac{\delta_i r_i}{8}\left\{\delta_i^2\left[r_i^2 \frac{d^3 f_0}{dr^3} - (2n_i + 1) r_i \frac{d^2 f_0}{dr^2} - n_i(3n_i + 2)\frac{df_0}{dr}\right]\right.$$

$$\left. + \delta_i\left[3r_i \frac{d^2\varphi_1}{dr^2} - (2n_i + 1)\frac{d\varphi_1}{dr}\right] + 4\left[2\frac{df_2}{dr} + \frac{d\varphi_2}{dr}\right]\right\},$$

$$\varphi_3(r_i,\alpha_i) = -\frac{\delta_i r_i}{24}\left\{\delta_i^2\left[r_i^2 \frac{d^3 f_0}{dr^3} + 3(2n_i + 1) r_i \frac{d^2 f_0}{dr^2} + 3n_i(3n_i + 2)\frac{df_0}{dr}\right]\right.$$

$$\left. + \delta_i\left[3r_i \frac{d^2\varphi_1}{dr^2} + 3(2n_i + 1)\frac{d\varphi_1}{dr}\right] + 12\frac{d\varphi_2}{dr}\right\},$$

TABLE 1. Temperature T/M Values at Points of the Square Plate ($n_1 = 3$, $\varepsilon = -1/9$)

Approximations	$\Theta = 0°$			$\Theta = 45°$		
	$1{,}5r_2$	$2r_2$	on L_1	$2{,}3r_2$	$2{,}5r_2$	on L_1
0	0,449	0,201	0,0477	0,105	0,076	−0,043
1	0,447	0,195	0,0235	0,118	0,094	0,034
2	0,443	0,187	0,0057	0,106	0,079	−0,011
3	0,4427	0,186	0,0016	0,108	0,082	0,0038

TABLE 2. Temperature T/M Values at Points of the Triangular Plate ($n_1 = 2$, $\varepsilon = 1/5$)

Approximations	$\Theta = 0°$				$\Theta = 60°$			
	$1{,}6r_2$	$2r_2$	$3r_2$	on L_1	$1{,}2r_2$	$1{,}6r_2$	$2r_2$	on L_1
0	0,248	0,406	1	1,669	0,091	0,248	0,406	0,594
1	0,197	0,293	0,488	0,568	0,106	0,299	0,518	0,811
2	0,264	0,403	0,771	1,063	0,108	0,366	0,628	0,974
3	0,260	0,394	0,733	0,980	0,109	0,370	0,637	0,991
4	0,262	0,397	0,747	1,016	0,110	0,372	0,640	0,997

TABLE 3. Temperature T/M Values at Points of the Elliptical Plate ($n_1 = 1$, $\varepsilon = 1/7$)

Approximations	$\Theta = 0°$			$\Theta = 90°$		
	$1{,}5r_2$	$1{,}8r_2$	on L_1	$1{,}2r_2$	$1{,}5r_2$	on L_1
0	0,271	0,079	—0,145	0,746	0,271	0,168
1	0,340	0,198	0,092	0,729	0,201	0,075
2	0,302	0,139	—0,037	0,662	0,163	0,025
3	0,315	0,162	0,013	0,659	0,150	0,008
4	0,311	0,155	—0,004	0,658	0,146	0,002

$$f_4(r_i) = -\frac{\delta_i r_i}{8}\left\{\frac{\delta_i^3}{8}\left[r_i^3\frac{d^4f_0}{dr^4} - 2(2n_i+1)r_i^2\frac{d^3f_0}{dr^3} + (4n_i+3)r_i\frac{d^2f_0}{dr^2}\right.\right.$$
$$\left. -(4n_i+3)\frac{df_0}{dr}\right] + \frac{\delta_i^2}{2}\left[r_i^2\frac{d^3\varphi_1}{dr^3} - (2n_i+1)r_i\frac{d^2\varphi_1}{dr^2} - (3n_i+2)n_i\frac{d\varphi_1}{dr}\right]$$
$$\left. + \delta_i\left[2r_i\frac{d^2f_2}{dr^2} + r_i\frac{d^2\varphi_2}{dr^2} - 2(2n_i+1)\frac{df_2}{dr} + (2n_i+1)\frac{d\varphi_2}{dr}\right] + 4\frac{d\Phi_3}{dr}\right\},$$
$$\Phi_4(r_i,\ \alpha_i) = -\frac{\delta_i^4}{48}r_i\left\{r_i^3\frac{d^4f_0}{dr^4} - 3\left[4n_i(n_i+1)+1\right]r_i\frac{d^2f_0}{dr^2}\right.$$
$$\left. -\left[4(n_i+1)(4n_i-1)n_i-3\right]\frac{df_0}{dr}\right\} - \frac{\delta_i^3}{12}r_i^3\frac{d^3\varphi_1}{dr^3} - \frac{\delta_i^2 r_i}{4}\left[r_i\frac{d^2f_2}{dr^2}\right.$$
$$\left. + r_i\frac{d^2\varphi_2}{dr^2} + (2n_i+1)\frac{df_2}{dr} - (2n_i+1)\frac{d\varphi_2}{dr}\right] - \frac{\delta_i r_i}{2}\left(\frac{d\Phi_3}{dr} + \frac{d\varphi_3}{dr}\right),$$
$$\varphi_4(r_i,\ \alpha_i) = -\frac{\delta_i^4 r_i}{192}\left\{r_i^3\frac{d^4f_0}{dr^4} + 6(2n_i+1)r_i^2\frac{d^3f_0}{dr^3} - \right.$$
$$\left. + \left[12n_i(4n_i+3)+3\right]r_i\frac{d^2f_0}{dr^2} + \left[16n_i(n_i+1)+3\right](4n_i-1)\frac{df_0}{dr}\right\}$$
$$-\frac{\delta_i^3 r_i}{48}\left[r_i^2\frac{d^3\varphi_1}{dr^3} + 3(2n_i+1)r_i\frac{d^2\varphi_1}{dr^2} + 3n_i(3n_i+2)\frac{d\varphi_1}{dr}\right]$$
$$-\frac{\delta_i^2 r_i}{8}\left[r_i\frac{d^2\varphi_2}{dr^2} + (2n_i+1)\frac{d\varphi_2}{dr}\right] - \frac{\delta_i r_i}{2}\cdot\frac{d\varphi_3}{dr},$$
$$i = 1,\ 2,\quad \delta_i = \begin{cases}1 & \text{for } i=1,\\ s & \text{for } i=2.\end{cases} \tag{6}$$

2. We will illustrate the temperature calculation on plates with $\beta = 93$ W/m$^2\cdot$°C, $\lambda = 58.15$ W/m$\cdot$°C, and $T_0 = 0$. We consider a square plate ($n_1 = 3$, $\varepsilon = -1/9$), a triangular plate ($n_1 = 2$, $\varepsilon = 1/5$), and an elliptical plate ($n_1 = 1$, $\varepsilon = 1/7$), all with a circular hole inside. These plates with their respective parameters are shown in Fig. 1. For the square and the elliptical plate we assume temperature T = 0 on the outside contour and T = M on the inside contour. For the triangular plate we assume temperature T = 0 on the inside contour and T = M on the outside contour. The dimensions are as follows: square plate $r_1 = 180$ mm, $r_2 = 60$ mm, and h = 8 mm; triangular plate $r_1 = 150$ mm, $r_2 = 50$ mm, and h = 8 mm; elliptical plate $r_1 = 140$ mm, $r_2 = 70$ mm, h = 8 mm. The values of temperature, within an accuracy to M, are given in

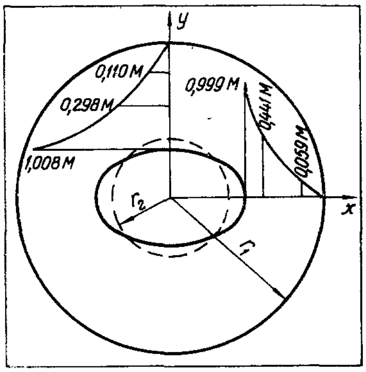

Fig. 2. Graphs of temperature variation along the x and y axes, for a plate whose inside contour is defined by Eqs. (1) and whose parameters are $r_i = r_2$, $\varepsilon_i = 1/7$, $n_i = 1$.

Tables 1-3 for various points of the plates. The accuracy of the results can be evaluated on the basis of the error allowed in satisfying the boundary conditions. Temperature T = M on one of the contours is taken as the 100% level. According to the tables, the largest error in the case of a square plate is 0.38%, in the case of a triangular plate is 1.6%, and in the case of an elliptical plate is 0.4%. Owing to the satisfactory convergence, the results for the square plate are carried to the third approximation. An analysis will show that the solution for a curvilinear outside contour converges faster as the ratio r_1/r_2 is increased.

The temperature variation along the x and the y axis is shown in Fig. 2 for a plate with a circular outside and an elliptical inside contour: $r_1 = 210$ mm, $r_2 = 70$ mm, h = 8 mm. The temperature here is T = M on the outside contour and T = 0 on the inside contour. The largest error in the solution in this case is 0.8%. The numerical computations here were made with the aid of the tables in [4, 5].

NOTATION

L_i	is the plate contour;
i	is the index number of the contour;
x, y	are the space coordinates;
r_i, n_i	are the parameters characterizing the dimensions and the shape of a contour;
Θ	is the parameter determining the position of a point on a curvilinear contour or the polar angle on a circular contour;
ε_i	is the parameter characterizing the deviation of contour L_i from a circular one;
M_1, M_2	are the values of the temperature on the outside and the inside contour, respectively;
T	is the temperature function;
k_i, m_i	are functions of the parameter Θ;
x_i, y_i	are the coordinates of a point on a circle with radius r_i;
f, φ, Φ	are functions of the radius r;
T_0, T_1, T_2, T_3, T_4	are functions defining the temperature distribution;
α_1, α_2	are parameters;
A_k, B_k, $C_{k\alpha}$, $D_{k\alpha}$, E_α, N_α, $F_{2\alpha}$, $R_{2\alpha}$	are integration constants;
β	is the heat transfer coefficient;
λ	is the thermal conductivity;
T_0	is the ambient temperature;
ν	is the function of β and λ;
I, K	are modified Bessel functions;
h	is the plate thickness.

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